3.6 Homomorphism

**Definition 3.27 Homomorphism**

Let $G$ be a group with respect to $\otimes$, and let $G'$ be a group with respect to $\oplus$. A *homomorphism* from $G$ to $G'$ is a mapping $\phi: G \rightarrow G'$ such that

$$\phi(x \otimes y) = \phi(x) \oplus \phi(y)$$

for all $x$ and $y$ in $G$.

**Endomorphism**

If $G = G'$, the homomorphism $\phi$ is an **endomorphism**.

**Epimorphism**

A homomorphism $\phi$ is called an epimorphism if this function is onto.

**Monomorphism**

If $\phi$ is one-to-one, then it is called a monomorphism.

**Example 1** Define $\phi: (\mathbb{Z},+) \rightarrow (\mathbb{Z}_n,\ast)$ by $\phi(x) = [x]$. Then for all $x$ and $y$ in $\mathbb{Z}$,

$$\phi(x + y) = [x + y] = [x] + [y] = \phi(x) + \phi(y).$$

Thus this function is a homomorphism.

**Endo? Epi? Mono?**

**Example 2** Define $\phi: G \rightarrow \widetilde{G}$ by $\phi(x) = \tilde{e}$ for all $x \in G$. Here $\tilde{e}$ is the identity element of $\widetilde{G}$.

Then for all $x$ and $y$ in $G$,

$$\phi(x) \phi(y) = \tilde{e} \tilde{e} = \tilde{e} = \phi(x) \phi(y).$$

Thus this function is a homomorphism.

**Endo? Epi? Mono?**
Example Consider the additive group \( \mathbb{Z} \) and the multiplicative group \( G = \{1, -1, -i, i\} \) and define \( \phi: \mathbb{Z} \to G \) by \( \phi(n) = i^n \). Prove that \( \phi \) is a homomorphism. Is this function epi and mono?

**Theorem 3.28 Images of Identities and Inverses**

Let \( \phi \) be a homomorphism from the group \( G \) to the group \( G' \). If \( e \) denotes the identity in \( G \), and \( e' \) denotes the identity in \( G' \), then

a. \( \phi(e) = e' \), and

b. \( \phi(x^{-1}) = [\phi(x)]^{-1} \) for all \( x \) in \( G \).

**Definition.** If there exists an epimorphism from the group \( G \) to the group \( \tilde{G} \), then \( \tilde{G} \) is called a homomorphic image.

**Definition 3.29 Kernel**

Let \( \phi \) be a homomorphism from the group \( G \) to the group \( G' \). The kernel of \( \phi \) is the set

\[
\ker \phi = \{ x \in G \mid \phi(x) = e' \}
\]

where \( e' \) denotes the identity in \( G' \).

**Example 5** To illustrate Definition 3.29, we list the kernels of the homomorphisms from the preceding examples in this section.

The kernel of the homomorphism \( \phi: \mathbb{Z} \to \mathbb{Z}_n \) defined by \( \phi(x) = [x] \) in Example 1 is given by

\[
\ker \phi = \{ x \in \mathbb{Z} \mid x = kn \text{ for some } k \in \mathbb{Z} \},
\]

since \( \phi(x) = [x] = [0] \) if and only if \( x \) is a multiple of \( n \).

The homomorphism \( \phi: \mathbb{Z} \to G \) in Example 3 defined by

\[
\phi(n) = \begin{cases} 
1 & \text{if } n \text{ is even} \\
-1 & \text{if } n \text{ is odd}
\end{cases}
\]

has the set \( \mathbb{E} \) of all even integers as its kernel, since 1 is the identity in \( G \).

For \( \phi: \mathbb{Z} \to \mathbb{Z} \) defined by \( \phi(x) = 5x \) in Example 4, we have \( \ker \phi = \{0\} \), since \( 5x = 0 \) if and only if \( x = 0 \). This kernel is an extreme case since part a of Theorem 3.28 assures us that the identity is always an element of the kernel.

At the other extreme, the homomorphism \( \phi: G \to G' \) defined in Example 2 by \( \phi(x) = e' \) for all \( x \in G \) has \( \ker \phi = G \).